JOURNAL OF APPROXIMATION THEORY 31, 299–324 (1981)

Approximation by Linear Combinations of Multivariate B-Splines

WOLFGANG DAHMEN

IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

Communicated by Charles A. Micchelli

Received August 4, 1980

This paper is concerned with the approximation of functions by linear combinations of multivariate B-splines. We construct and analyze local linear approximation schemes on certain uniform configurations. Furthermore we point out how these uniform configurations may be refined locally while still preserving the desired global smoothness of the splines.

1. INTRODUCTION

This paper is concerned with an attempt to exploit the advantageous properties of multivariate B-splines (cf. [4, 10, 19]) for the purpose of approximation. One attractive property of multivariate B-splines is the fact that for an arbitrary spatial dimension s and any degree k a B-spline generally belongs to $C^{k-1}(R^s)$ (the space of functions possessing continuous partial derivatives of order $k-1$) as well as being locally supported. This fact should be contrasted with the many examples in the literature (cf., e.g., [7, 20, 22]) which affirm that being a multivariate piecewise polynomial of high global smoothness (compared to its degree) and at the same time having local support are usually conflicting properties. Of course, both properties are important for various applications such as smooth surface fitting or solving higher order boundary value problems by finite element methods.

At first we shall briefly discuss a rather general class of spline spaces for arbitrary spatial dimension s and any degree k introduced in [11]. Since these spaces are actually linear spans of appropriately selected B-splines, their elements possess $k-1$ continuous derivatives. Moreover, although our own piecewise polynomials are only of total degree k we shall show that they provide the same approximation rates achieved by tensor product constructions, which use coordinatewise degree.

Our main concern in this paper is to focus upon an important subclass of the spline spaces in which certain "uniform" configurations of knot sets are

0021-9045/81/040299–26$02.00/0

chosen. This allows us to obtain local linear approximation schemes similar to those considered in [5, 18] for the univariate case. Moreover, these uniform configurations which we propose give us the facility to make local refinements without interfering with the required global smoothness. So, the considerable practical advantage of uniform grids can be combined with the profit of "local adaptation" (cf. [2, 14]). This may provide an interesting method for finding smooth multivariate adaptive schemes which are generally hard to construct. (cf. the final remarks in [6]).

Let us fix now some notation which will be used throughout this paper. $\mathbf{x}$, $\mathbf{z}$ will usually be elements of the Euclidean space R^s with components x_i which we sometimes also write as $(\mathbf{x})_i$. We will always consider $\mathbf{x}$ as a column vector and its transpose is denoted by $\mathbf{x}^T$. For $\mathbf{a}, \mathbf{b} \in R^s$ we briefly write $\mathbf{ab} = (a_1 b_1 \cdots a_s b_s)^T$, $\mathbf{a}/\mathbf{b} = (a_1/b_1 \cdots a_s/b_s)^T$ for $c \in R$, $\mathbf{a}^c = (a_1^c \cdots a_s^c)^T$ and $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^s a_i^{b_i}$. Furthermore, we set $[\mathbf{a}, \mathbf{b}] = \{\mathbf{u} \in R^s : a_i \leqslant u_i \leqslant b_i,\ i = 1, 2,..., s\}$. In particular, $[0, 1]^s$ is the unit s-cube. For $\mathbf{z}_0,..., \mathbf{z}_s \in R^s$ we set

$$\det \begin{pmatrix} \mathbf{z}_0 & \cdots & \mathbf{z}_s \\ 1 & \cdots & 1 \end{pmatrix} = \begin{vmatrix} (\mathbf{z}_0)_1 & \cdots & (\mathbf{z}_s)_1 \\ \vdots & & \vdots \\ (\mathbf{z}_0)_s & \cdots & (\mathbf{z}_s)_s \\ 1 & \cdots & 1 \end{vmatrix}$$

Multi-indices are denoted by $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{v} \in Z_+^s$ equipped with the norm $|\boldsymbol{\alpha}| = \sum_{i=1}^s \alpha_i$. In particular, we briefly write $\mathbf{1} = (1 \cdots 1)^T \in R^s$. We use $\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}$ to mean what $\beta_i \leqslant \alpha_i$, and also set $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \boldsymbol{\alpha}!/\boldsymbol{\beta}!(\boldsymbol{\alpha} - \boldsymbol{\beta})!$, where $\boldsymbol{\alpha}! = \prod_{i=1}^s \alpha_i!$.

For any domain $\Omega \subset R^s$, $\Pi_k(\Omega) = \{\sum_{|\boldsymbol{\alpha}| \leqslant k} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} : c_{\boldsymbol{\alpha}} \in R,\ \mathbf{x} \in \Omega\}$ is the space of polynomials of "total degree" k on Ω. $\|\cdot\|_p(\Omega)$ denotes the usual L_p-norm taken with respect to the domain $\Omega \subset R^s$ with the familiar interpretation in case $p = \infty$. Furthermore, denoting by $D^{\boldsymbol{\alpha}} f$ the partial derivatives of order $\boldsymbol{\alpha}$ of the function f, the Sobolev spaces of order k are given by $W_p^k(\Omega) = \{f : \|D^{\boldsymbol{\alpha}} f\|_p \in L_p(\Omega), |\boldsymbol{\alpha}| \leqslant k\}$, where again $W_\infty^k(\Omega) = C^k(\Omega)$.

We shall also use the following notation for the directional derivative of f along $\mathbf{z} \in R^s$, namely, $D_{\mathbf{z}} f = \sum_{i=1}^s z_i (\partial f / \partial x_i)$.

Finally, C will denote a generic constant which may take different values at each occurrence and $[\Gamma]$ is used to denote the convex hull of a given set $\Gamma \subset R^s$.

2. Some Properties of Multvariate B-Splines

In this section we list some properties of the multivariate B-spline which play a central role throughout this paper.

Just as in the univariate case an s-dimensional B-spline is defined for every knot set $P = \{\mathbf{x}_0, ..., \mathbf{x}_n\} \subset R^s$, $n \geqslant s$, with

$$\dim \operatorname{span}([P]) = s. \tag{2.1}$$

We postpone te definition of the B-spline but instead remark that for $n = s$ we can represent it by

$$\begin{aligned} M(\mathbf{x} \mid P) &= s! \left| \det \begin{pmatrix} \mathbf{x}_0 \cdots \mathbf{x}_s \\ 1 \cdots 1 \end{pmatrix} \right|^{-1}; \qquad && \mathbf{x} \in [P], \\ &= 0; && \mathbf{x} \notin [P]. \end{aligned} \tag{2.2a}$$

So, $M(\mathbf{x} \mid P)$ is then the characteristic function of the simplex $[P]$ divided by the volume of $[P]$. In general, when $n > s$, we have the recurrence relation

$$M(\mathbf{x} \mid P) = \frac{n}{n-s} \sum_{j=0}^{s} C_{i_j}(\mathbf{x} \mid \mathbf{x}_{i_0} \cdots \mathbf{x}_{i_s})\, M(\mathbf{x} \mid P \backslash \{\mathbf{x}_{i_j}\}), \qquad \mathbf{x} \in R^s, \tag{2.2b}$$

whenever $\mathbf{x}_{i_i} \in P$ are chosen so that $\operatorname{vol}_s([\{\mathbf{x}_{i_0} \cdots \mathbf{x}_{i_s}\}]) > 0$ and hence the barycentric coordinates

$$C_{i_j}(\mathbf{x} \mid \mathbf{x}_{i_0} \cdots \mathbf{x}_{i_s}) := \det \begin{pmatrix} \mathbf{x}_{i_0} \cdots \mathbf{x}_{i_{j-1}}, & \mathbf{x}, & \mathbf{x}_{i_{j+1}} \cdots \mathbf{x}_{i_s} \\ 1 \cdots 1 & 1 & 1 \cdots 1 \end{pmatrix} \Big/ \det \begin{pmatrix} \mathbf{x}_{i_0} \cdots \mathbf{x}_{i_s} \\ 1 \cdots 1 \end{pmatrix} \tag{2.3}$$

are well defined.

These formulae were first stated by Micchelli [19]. An alternative approach is given in [6]. Since by (2.2a, b) and (2.3) a B-spline is at any point a convex combination of lower order B-splines, one can show that the above relations give rise to stable algorithms for the numerical evaluation of B-splines (cf. [8, 13]), i.e., the relative round off errors can be bounded by the accuracy of the computer times a constant which depends only on s and n but not on the position of the knots.

The above relations (2.2a, b), (2.3) also affirm that $M(\mathbf{x} \mid P)$ is a polynomial of total degree $n - s$ within any region which is enclosed but not intersected by any $(s-1)$-simplex spanned by elements of P.

Furthermore, the smoothness of $M(\mathbf{x} \mid P)$ is related to the configuration of P as follows: $M(\mathbf{x} \mid P) \in C^{n-s-d}(R^s)$ if every $s + d$ elements of P span an s-dimensional set (cf. [10, 19]). In particular, in the terminology of [10]

$$M(\mathbf{x} \mid P) \in C^{n-1-s}(R^s) \tag{2.4}$$

if P is "0-degenerate," i.e., every $s + 1$ knots in P are affinely independent. The derivatives of $M(\mathbf{x} \mid P)$ (whenever they exist) are, as in the univariate case, linear combinations of lower order B-splines (cf. [10, 19]).

$$D_{\mathbf{z}} M(\mathbf{x} \mid P) = n \sum_{j=0}^{s} c_j M(\mathbf{x} \mid P \backslash \{\mathbf{x}_{i_j}\}), \tag{2.5}$$

where

$$\sum_{j=0}^{s} c_j \mathbf{x}_{i_j} = \mathbf{z}, \qquad \sum_{j=0}^{s} c_j = 0. \tag{2.6}$$

Thus by Cramer's rule $c_j = D_{\mathbf{z}} C_{i_j}(\mathbf{x} | \mathbf{x}_{i_0} \cdots \mathbf{x}_{i_s})$.

The following geometric interpretation of the B-spline which usually serves as its definition will play a crucial role throughout this paper (cf. [4]).

To any n-simples $\sigma = [\{\mathbf{v}_0,..., \mathbf{v}_n\}]$ with vertices $\mathbf{v}_i$ one may assign a knot set $P(\sigma) = \{\mathbf{x}_0,..., \mathbf{x}_n\} \subset R^s$ by setting

$$\mathbf{x}_i = \mathbf{v}_i |_{R^s}. \tag{2.7}$$

Note that $P(\sigma)$ satisfies (2.1) if $\operatorname{vol}_n(\sigma) > 0$. The following identity holds for any non-degenerate n-simples σ (cf. [11, 19])

$$\operatorname{vol}_n(\sigma)\, M(\mathbf{x} \mid P(\sigma)) = M_\sigma(\mathbf{x}), \tag{2.8}$$

where

$$M_\sigma(\mathbf{x}) = \operatorname{vol}_{n-s}(\{\mathbf{u} \in \sigma \colon \mathbf{u} |_{R^s} = \boldsymbol{x}\}). \tag{2.9}$$

This confirms that $M(\mathbf{x} \mid P)$ is supported on $[P]$.

Estimating the derivatives of the remainder in approximating a given function by linear combinations of B-splines (cf. Section 5) is essentially reduced by (2.5) to estimating the norms of the B-spline. One can actually show that [12]

$$\begin{aligned} \operatorname{vol}_s([P])^{-1+1/p} &\leqslant \|M(\cdot \mid P)\|_p \\ &\leqslant \frac{n!}{s!} \binom{n+1}{s+1} \operatorname{vol}_s([P])^{-1+1/p}, \qquad 1 \leqslant p \leqslant \infty, \end{aligned} \tag{2.10}$$

by constructing a simplex σ such that $P = P(\sigma)$, $\sigma \subset [P] \times [0, 1]^{n-s}$ as well as $\operatorname{vol}_n(\sigma) \geqslant \frac{s!}{n!} \binom{n}{s}^{-1} \operatorname{vol}_s([P])$ and using (2.8). As an immediate consequence, we get

$$\begin{aligned} \|D_{\mathbf{z}} M(\cdot \mid P)\|_p \leqslant \frac{n!}{s!} (s+1) \binom{n-1}{s} \\ \times \min_I \{ \max_{j=0,\dots,s} |D_{\mathbf{z}} C_{i_j}(\mathbf{x} \mid \mathbf{x}_{i_0},..., \mathbf{x}_{i_s})| \operatorname{vol}_s([P \backslash \{\mathbf{x}_{j_s}\}])^{-1+1/p} \}, \end{aligned} \tag{2.11}$$

where $I = \{\{i_0,..., i_s\} \subset \{0,..., n\} \colon \operatorname{vol}_s([(\mathbf{x}_{i_0},..., \mathbf{x}_{i_s}\}]) > 0\}$.

3. Linear Combinations of Multivariate B-Splines

The previously stated properties of B-splines motivate us to develop methods for approximating functions by linear combinations of these B-splines. To this end, the first essential task is to define a linear space of B-splines which has an appropriate structure for approximation. Of course, "appropriate" should mean that polynomials of highest degree are to be contained in this spline space. It is shown in [11] that this can be realized by exploiting the geometric interpretation (2.8), (2.9). Indeed, following a suggestion of de Boor [4] we define for a bounded domain $\Omega \subset R^s$ the "cylinder" $\Omega_{n,s} = \Omega \times [0,1]^{n-s}$ and consider a triangulation $\mathscr{E}(\Omega_{n,s}) = \{\sigma_i\}_{i=1}^N$ of $\Omega_{n,s}$. Here a collection $\mathscr{E}(\Gamma)$ of simplices is called a triangulation of Γ if Γ is contained in the union of these simplices and if the intersection of any two of these simplices is either empty or exactly one common lower dimensional face.

This construction implies, on account of (2.7) and (2.8), that

$$\sum_{i=1}^{N} M_{\sigma_i}(\mathbf{x}) = 1, \qquad \mathbf{x} \in \Omega. \tag{3.1}$$

Setting

$$\sigma_i = [\{\mathbf{v}_0^i, \ldots, \mathbf{v}_n^i\}], \qquad P_i = P(\sigma_i), \qquad \mathscr{P} = \{P_i : i = 1, \ldots, N\} \tag{3.2}$$

then (3.1) means, in view of (2.7), that the linear span

$$\mathscr{S}_k(\mathscr{P}, \Omega) := \operatorname{span}\{M(\mathbf{x} \mid P) : P \in \mathscr{P}, \mathbf{x} \in \Omega\} \tag{3.3}$$

contains the constant functions on Ω. Here the integer k will always denote the difference $n - s$ and hence the degree of the splines. Let us state now an immediate consequence of (3.3) concerning the approximation by elements of $\mathscr{S}_k(\mathscr{P}, \Omega)$. Fixing $\boldsymbol{\tau}_i \in [P_i]$, $i = 1, \ldots, N$, one clearly has

$$\begin{aligned} \operatorname{dist}_\infty(f : \mathscr{S}_k(\mathscr{P}, \Omega)) &:= \inf_{S \in \mathscr{S}_k(\mathscr{P}, \Omega)} \|f - S\|_\infty \\ &\leqslant \left\| f(\mathbf{x}) - \sum_{i=1}^{N} f(\boldsymbol{\tau}_i) M_{\sigma_i}(\mathbf{x}) \right\|_\infty \end{aligned}$$

From (3.1) and the fact that $M_{\sigma_i}(\mathbf{x})$ is supported on $[P_i]$ one may easily derive (cf. [3, 11]) that

$$\operatorname{dist}_\infty(f : \mathscr{S}_k(\mathscr{P}, \Omega)) \leqslant \omega(f, h, \Omega), \tag{3.4}$$

where $h := \max\{\operatorname{diam}([P]) : P \in \mathscr{P}\}$ and $\omega(f, h, \Omega) := \sup\{|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})| : \mathbf{x}, \mathbf{x} + \mathbf{y} \in \Omega, \|\mathbf{y}\| \leqslant h\}$ is the usual modulus of continuity, relative to the Euclidean norm $\|\cdot\|$ on R^s.

Next we will point out how one may improve the estimate (3.4) when approximating smooth functions on some bounded domain Ω contained in $[0, 1]^s$. Let $\boldsymbol{\alpha} \in Z_+^s$, $|\boldsymbol{\alpha}| \leqslant k$, $\alpha_0 = 0$ and $I = \{i_1, \dots, i_{|\boldsymbol{\alpha}|}\} \subset \{s+1, \dots, n\}$ be some set of distinct (but not necessarily increasing) indices, i.e., $|I| = |\boldsymbol{\alpha}|$. Let $a_j := \sum_{i=0}^{j} \alpha_i$. We define for $t \in [0, 1]$ and every I, $|I| = |\boldsymbol{\alpha}|$, the map $F_t(\cdot, I)\colon R^n \to R^n$ by

$$\begin{aligned}(F_t(\mathbf{u}; I))_i &= u_i; && i \notin I,\\ &= u_i(1 - t(1 - u_j)); && i \in \{i_{a_{j-1}+1}, \dots, i_{a_j}\} \subset I,\\ & \quad j = 1, \dots, s.\end{aligned} \tag{3.5}$$

Setting

$$\sigma_i(t, I) := [\{F_t(\mathbf{v}_0^i; I), \dots, F_t(\mathbf{v}_n^i; I)\}] \tag{3.6}$$

one can show by the same arguments which were used in [11] for a special choice of I that there is for every $\mathscr{P}$ given by (3.2) $t_0 > 0$ so that for $t \in [0, t_0]$ and $|I| = |\boldsymbol{\alpha}| \leqslant k$

$$(\mathbf{1} - t(\mathbf{1} - \mathbf{x}))^{\boldsymbol{\alpha}} = \sum_{i=1}^{N} \operatorname{vol}_n(\sigma_i(t; I)\, M(\mathbf{x} \mid P_i) \qquad \text{for} \quad \mathbf{x} \in \Omega. \tag{3.7}$$

Note that (3.1) is obtained by choosing $t = 0$ or $\boldsymbol{\alpha} = \mathbf{0}$.

The inclusion

$$\Pi_k(\Omega) \subset \mathscr{S}_k(\mathscr{P}, \Omega) \tag{3.8}$$

is now an immediate consequence of (3.7). This fact will be exploited in the following sections to construct linear local approximation schemes for certain uniform configurations of knot sets providing optimal approximation rates.

Finally, one should note that in principle the global structure of these spline spaces is by contruction not affected by "local modifications" such as refinements (of the underlying triangulation). This fact is in contrast to tensor product spaces.

4. Uniform Configurations of Knot Sets

In this section we shall deal with a special case of the rather general construction (3.2), (3.3). This concrete construction may be viewed as a first attempt to give instances of the multivariate spline spaces $\mathscr{S}_k(\mathscr{P}, \Omega)$ which may be used for numerical calculations.

The idea is to make up special triangulations of $R^s \times [0, 1]^k$ by first decomposing $R^s \times [0, 1]^k$ into congruent parallelepipeds which are in turn triangulated in a canonical way described below.

Let the vectors $\mathbf{e}_i = (\delta_{ij})_{j=1}^n$ form the usual standard orthonormal base in R^n. Denoting by Per_n the group of permutations of the set $\{1,..., n\}$ one may assign to any $\pi \in \mathrm{Per}_n$ an n-simplex $\sigma_\pi = [\{\mathbf{v}_0^\pi,..., \mathbf{v}_n^\pi\}]$ by

$$\mathbf{v}_0^\pi = 0, \qquad \mathbf{v}_j^\pi = \mathbf{v}_{j-1}^\pi + \mathbf{e}_{\pi(j)}, \qquad j = 1,..., n. \tag{4.1}$$

$\mathscr{K}_n = \{\sigma_\pi : \pi \in \mathrm{Per}_n\}$ is called "Kuhn's triangulation" of the unit cube $[0, 1]^n$ (cf. [1, 17]). When A is the affine transformation which takes $[0, 1]^n$ into a parallelepiped V Kuhn's triangulation of V is given by $\mathscr{K}_n(V) = \{A(\sigma_\pi): \pi \in \mathrm{Per}_n\}$.

In order to illustrate the structure of $\mathscr{K}_n$ and for later reference, we list some known properties of these triangulations (cf. [1, 17]).

PROPOSITION 4.1. *Let $\mathscr{K}_n$ be defined by* (4.1).

(i) *All elements of $\mathscr{K}_n$ have equal volume.*

(ii) *The restriction of $\mathscr{K}_n$ to any r-face Q of $[0, 1]^n$ coincides with $\mathscr{K}_r(Q)$.*

(iii) *$\mathscr{K}_n$ is "compatible with translations," i.e., the induced triangulation of any $(n-1)$ face of $[0, 1]^n$ can be obtained by translating the triangulation of the opposite $(n-1)$-face.*

(iv) *$\mathscr{K}_n$ is sequential, i.e., assigning a graph $G(\mathscr{K}_n)$ by taking $\sigma_\pi \in \mathscr{K}_n$ as nodes where two nodes are connected if the corresponding simplices have a nonempty common $(n-1)$-face, then "sequential" means that $G(\mathscr{K}_n)$ can be traversed by passing through each node precisely once.*

Note that by (iii) the union of Kuhn's triangulation of two adjacent cubes having one $(n-1)$-face in common is a triangulation of the union of both cubes.

Of course, the same statements are valid with respect to $\mathscr{K}_n(V)$, where V is any parallelepiped in R^n.

When writing $\mathbf{x} + \mathbf{u}$ or $\mathbf{xu}$, where $\mathbf{x} \in R^s$ and $\mathbf{u} \in R^n$, $\mathbf{x}$ will be always understood to be raised to an n-vector by setting $x_i = 0$, $i = s+1,..., n$. In contrast to this $\hat{\mathbf{x}}$ denotes for any $\mathbf{x} \in R^s$ the n-vector $(x_1 \cdots x_s \; 1 \cdots 1)^T$.

The second ingredient of our construction is an $(n \times n)$-matrix of type

$$A = \begin{pmatrix} 1 \cdots 0 & a_{11} \cdots a_{1k} \\ \vdots \quad\; \vdots & \vdots \qquad \vdots \\ 0 \cdots 1 & a_{s1} \cdots a_{sk} \\ & 1 \qquad 0 \\ \vdots & \qquad \ddots \;\; \vdots \\ 0 \;\; \cdots & \cdots \quad \cdots 1 \end{pmatrix}. \tag{4.2}$$

In fact, setting $Q=[0,1]^n$, A gives rise for any $\mathbf{h} \in R^s_+$ to the following collection of parallelepipeds:

$$\mathscr{g}_{\mathbf{v}} = \mathscr{g}_{\mathbf{v}}(\mathbf{h}, A) = \hat{\mathbf{h}}(A(Q)+\mathbf{v}) = \{\hat{\mathbf{h}}\mathbf{u} + \hat{\mathbf{h}}\mathbf{v}, \mathbf{u} \in A(Q)\}, \qquad \mathbf{v} \in Z^s_+,$$

which are in turn used to define for any domain $\Omega \in R^s$ the spline spaces

$$\mathscr{S}_{k,\mathbf{h}}(A, \Omega) = \operatorname{span}\{M(\mathbf{x} \mid P(\sigma)) : \sigma \in \mathscr{K}_n(\mathscr{g}_{\mathbf{v}}(\mathbf{h}, A)), \mathbf{v} \in Z^s, [P(\sigma)] \cap \Omega \neq \varnothing\}. \tag{4.3}$$

In order to verify that $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ has good approximation properties we have to check first whether $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ is of the type (3.2). Indeed, by (4.2) and the definition of $\hat{\mathbf{h}}$ we clearly have

$$\bigcup \{\mathscr{g}_{\mathbf{v}} : \mathbf{v} \in Z^s\} = R^s \times [0,1]^k$$

and Proposition 4.1(iii) assures that

$$\mathscr{E}_{\mathbf{h},A} := \bigcup \{\mathscr{K}_n(\mathscr{g}_{\mathbf{v}}(\mathbf{h}, A)) : \mathbf{v} \in Z^s\}$$

is a triangulation of $R^s \times [0,1]^k$. Thus $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ is actually of the type (3.2).

In order to avoid that the supports of the B-splines are enlarged too much by A we introduce a further condition on the matrix A. Setting for any $a \in R$ $a^{\pm} = \max\{0, \pm a\}$, let

$$a_i^{\pm} := \sum_{j=1}^{k} a_{ij}^{\pm}, \mathbf{a}^{\pm} = (a_i^{\pm})_{i=1}^{s}. \tag{4.4}$$

We will always assume that

$$b := \max\{a_i^{\pm} : \pm, i = 1,..., s\} < \tfrac{1}{2}. \tag{4.5}$$

holds.

LEMMA 4.1. *Using the above notation* (4.5) *implies*

(i) $\operatorname{diam}([P(\sigma)]) \leqslant 2(\sqrt{s})h$ *for all* $\sigma \in \mathscr{E}_{\mathbf{h},A}$, *where* $h = \max\{h_i : i = 1,..., s\}$.

(ii) $1(A(Q)) := \{\mathbf{x} \in A(Q)|_{R^s} : \operatorname{vol}_k(\{\mathbf{u} \in A(Q) : \mathbf{u}|_{R^s} = \mathbf{x}\}) = 1\} \neq \varnothing$.

(iii) *Let* $\sigma_i \in \mathscr{K}_n(\mathscr{g}_{\mathbf{v}_i}(\mathbf{h}, A))$, $i = 1, 2$; *then*

$$[P(\sigma_i)] \cap [P(\sigma_2)] \neq \varnothing$$

holds only if $\mathscr{g}_{\mathbf{v}_1}(\mathbf{h}, A)$ and $\mathscr{g}_{\mathbf{v}_2}(\mathbf{h}, A)$ *are neighbors.*

Proof. Let $\mathbf{u}$ be any vertex of Q which is orthogonal to R^s. By the definition of b and (4.5) we have

$$\max\{|(A\mathbf{u})_i|: i=1,..., s\} \leqslant b < \tfrac{1}{2}. \tag{4.6}$$

Observing the definition of $\varphi_{\mathbf{v}}(\mathbf{h}, A)$ this readily implies (ii) and (iii). Moreover, (4.5) assures that $\operatorname{diam}(A(Q)|_{R^s}) \leqslant 2\sqrt{s}$ which establishes (i).

As for the practical construction of the splines in $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ consider the following collection of knot sets:

$$\begin{aligned} P_{\pi,\mathbf{v}} &= \{\mathbf{h}((A\mathbf{v}_j^\pi)\,|_{R^s} + \mathbf{v}): j=0,..., n\}, \qquad \mathbf{v} \in Z^s, \\ \mathscr{P}_{\mathbf{h},A} &= \{P_{\pi,\mathbf{v}};\, \pi \in \operatorname{Per}_n, \mathbf{v} \in Z^s\}, \end{aligned} \tag{4.7}$$

where $\sigma_\pi = [\{\mathbf{v}_0^\pi,..., \mathbf{v}_n^\pi\}] \in \mathscr{K}_n$. Clearly, (4.3) says that $\mathscr{S}_{k,\mathbf{h}}(A, \Omega) = \operatorname{span}\{M(\mathbf{x} \mid P): P \in \mathscr{P}_{\mathbf{h},A}, \mathbf{x} \in \Omega\}$.

According to the properties of the B-splines listed in Section 2, $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ consists of piecewise polynomials of total degree k. More precisely, these splines are polynomials in every region which is bounded but not intersected by any $(s-1)$-simples spanned by knots in $P \in \mathscr{P}_{\mathbf{h},A}$. Figures 4.1, 4.2, and 4.3 illustrate this for the case $s=k=2$, i.e., $n=4$ where the following matrix A gives rise to continuously differentiable splines.

$$A = \begin{pmatrix} 1 & 0 & -0.2 & 0.45 \\ 0 & 1 & 0.4 & -0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Figure 4.1 shows a typical knot set arising from projecting the vertices of a corresponding simplex in $\mathscr{K}_4$, whereas the configuration which is induced by all the elements of $\mathscr{K}_4$ is depicted in Fig. 4.2. Shifting this "basic unit" generates the whole configuration $\mathscr{P}_{\mathbf{h},A}$ a typical section of which (consisting of four basic units) is given in Fig. 4.3.

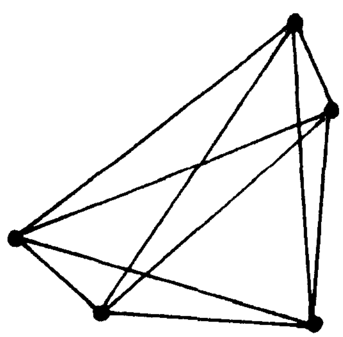

FIGURE 4.1

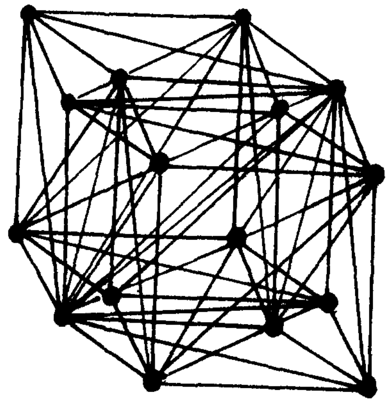

FIGURE 4.2

Recall that the structure of $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ is completely determined by the knot sets only. The corresponding "cut regions" shown in the figures are produced automatically. The practical construction of the knot sets amounts, according to (4.6), simply to projecting and shifting the vertices of the corresponding fixed simplices $\sigma_\pi \in \mathscr{K}_n$ (cf. (4.1)). (4.1) in turn may be performed for every $n \in N$ by efficient algorithms which exploit the sequential structure of $\mathscr{K}_n$ (cf. Proposition 4.1(iv)), e.g., by "reflections" (cf. [1]).

Accordingly, evaluating the elements of $\mathscr{S}_{k,\mathbf{h}}(A, \Omega)$ is reduced to evaluating only the B-splines (up to rescaling and shifting) which are induced by $\mathscr{K}_n$ (cf. Fig. 4.2).

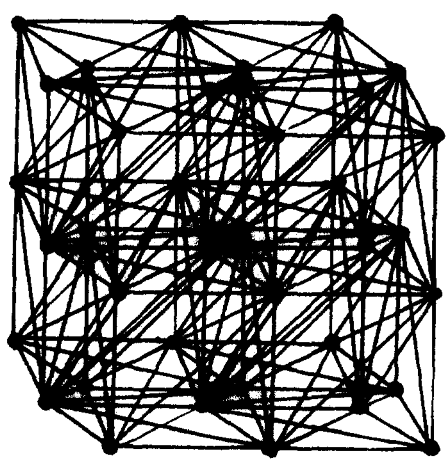

FIGURE 4.3

Now we will discuss in what way the global smoothness of the splines on $\mathscr{S}_{k,\mathbf{h}}(A,\Omega)$ is related to A. Starting with the simplest case where $a_{ij}=0$, $i=1,\ldots,s$, $j=1,\ldots,k$ in (4.2), we observe that $[P_{\pi,\mathbf{0}}]\in\mathscr{K}_s([0,h_1]\times\cdots\times[0,h_s])$. So $\mathscr{S}_{k,\mathbf{h}}(A,\Omega)$ is the space of piecewise polynomials of total degree k with respect to the triangulation $\bigcup\{[P_{\pi,\mathbf{v}}]:\mathbf{v}\in Z^s,\pi\in\mathrm{Per}_s\}$ without any smoothness constraints when passing from one simplex to a neighboring one.

The other extreme case of highest smoothness is more interesting and will turn out to be related to the following type of matrices A. To this end, consider the following collection of sets of disjoint sets of indices from 1 to n:

$$\mathscr{T}=\{J=\{I_i:i=1,\ldots,s\}:I_i\subset\{1,\ldots,n\},I_i\cap I_j=\varnothing,\ i\neq j\}.$$

Denoting by $\mathbf{A}_i$, $i=1,\ldots,n$, the columns of A restricted to R^s let

$$B_J=\left\{\sum_{j\in I_i}\mathbf{A}_j:i=1,\ldots,s\right\}.$$

The matrix A is called "dispersive" if one has for all $J\in\mathscr{T}$

$$\dim\operatorname{span}\{B_J\}=s, \tag{4.8}$$

i.e., every s sums over disjoint sets of projected columns of A are linearly independent.

A sufficient condition for A to be dispersive is, e.g., that the first s rows of A form a strictly sign consistent matrix of order s (cf. [16, p. 47]). This may be easily confirmed by evaluating $\det(B_J)$ so that one obtains a sum of s-minors of A. Since by definition all the summands have the same non-zero sign, we trivially get $\det(B_J)\neq 0$. As for examples of sign consistent matrices see also [16].

THEOREM 4.1. *Let* $\mathscr{S}_{k,\mathbf{h}}(A,\Omega)$ *be defined by* (4.3); *then*

$$\mathscr{S}_{k,\mathbf{h}}(A,\Omega)\subset C^{k-1}(\Omega)$$

holds if and only if $\mathrm{diag}_n(\hat{\mathbf{h}})A$ *is dispersive.*

Proof. The proof is based on the following characterization of the set $\mathscr{T}$. Let Λ_j^π denote the set of indices l for which the components $(\mathbf{v}_j^\pi)_l=1$, where $\mathbf{v}_j^\pi$ is again defined by (4.1). Observing that in view of (4.1) the $\mathbf{v}_j^\pi$ are lexicographically ordered, i.e., $(\mathbf{v}_j^\pi)_l\leqslant(\mathbf{v}_{j+1}^\pi)_l$, $j=0,\ldots,n-1$, $l=1,\ldots,n$, we conclude that

$$\Lambda_j^\pi\subset\Lambda_{j+1}^\pi,\qquad j=0,\ldots,n-1.$$

Furthermore, we observe that for any set of increasing indices $\{j_0,..., j_s\} \subset \{0,..., n\}$ and any $\pi \in \mathrm{Per}_n$ the collection $\{\Lambda_{j_i}^\pi \backslash \Lambda_{j_{i-1}}^\pi : i = 1,..., s\}$ belongs to $\mathscr{T}$. Moreover, it is not hard to see that all the elements of $\mathscr{T}$ are of this type. In fact, setting for $J_0 = \{I_j: j = 1,..., s\}$, $\Lambda_j = \bigcup \{I_r: r = 1,..., j\}$ (4.1) affirms the existence of $\sigma_{\pi_0} \in \mathscr{K}_n$ and $0 = r_0 < r_1 < \cdots < r_s \leqslant n$ such that $\Lambda_{r_0}^{\pi_0} = \varnothing$ and $\Lambda_{r_j}^{\pi_0} = \Lambda_j, j = 1,..., s$. Hence we have

$$\mathscr{T} = \{\{\Lambda_{j_i}^\pi \backslash \Lambda_{j_{i-1}}^\pi : i = 1,..., s\}: \pi \in \mathrm{Per}_n, \{j_0,..., j_s\} \subset \{0,..., n\}\}. \tag{4.9}$$

On account of (2.4) and (4.3) it is now sufficient to show that $\bar{A} = \mathrm{diag}_n(\bar{\mathbf{h}})A$ is dispersive iff $\{\bar{A}\mathbf{v}_j^\pi |_{R^s}: j = 0,..., n\}$ is 0-degenerate for all $\pi \in \mathrm{Per}_n$. To this end, note that

$$\bar{A}\mathbf{v}_j^\pi |_{R^s} = \sum_{l \in \Lambda_j^\pi} \bar{\mathbf{A}}_l, \tag{4.10}$$

where $\bar{\mathbf{A}}_l$ denotes the restriction of the lth column of $\bar{A}$ to R^s. So, by virtue of (4.10) we can write for any $\{j_0,..., j_s\} \subset \{0,..., n\}$

$$\sum_{l=1}^{s} c_l \bar{A}(\mathbf{v}_{j_l}^\pi - \mathbf{v}_{j_0}^\pi)|_{R^s} = \sum_{l=1}^{s} \left(\sum_{r=1}^{s+1-l} c_r\right)\left(\sum_{m \in \Lambda_{j_l}^\pi \backslash \Lambda_{j_{l-1}}^\pi} \bar{\mathbf{A}}_m\right).$$

Therefore, the homogeneous system

$$\sum_{l=1}^{s} c_l \bar{A}(\mathbf{v}_{j_l}^\pi - \mathbf{v}_{j_0}^\pi)|_{R^s} = 0$$

has for any set $\{j_0,..., j_s\} \subset \{0,..., n\}$, $\pi \in \mathrm{Per}_n$ only the trivial solution iff the respective set of vectors $(\sum_{m \in \Lambda_{j_l}^\pi \backslash \Lambda_{j_{l-1}}^\pi} \bar{\mathbf{A}}_m)$ are linearly independent which in view of (4.8) and (4.9) is the assertion.

If will turn out below that it is desirable to have such dispersive matrices which distribute the knots as uniformly as possible, that is to say, the ratios of the radii of the inscribed and circumscribed balls of any s-simplex spanned by $s+1$ vectors in any knot set are to be as large as possible. This may be realized by optimizing the entries a_{ij} of A subject to the restriction (4.5). Since for $k = 0$, the matrix $A = \mathrm{diag}_s(\mathbf{h})A$ is trivially dispersive, we may take it as a starting point for the recursive construction of higher order dispersive matrices. Clearly when appending an additional column to any dispersive matrix, the resulting matrix is not dispersive only on a set of measure zero.

We wish to distinguish between two kinds of information associated with every knot set which determines the structure of $\mathscr{S}_{\mathbf{h},A}$. On one hand the position of the knots and on the other hand their memberships to certain knot sets. By preserving the knot set structure but varying the position of the knots (within certain ranges) one obtains "distorted" configurations which may be viewed as counterparts to irregularly spaced knots in the univariate case. Since these configurations are generated by appropriate distortions of the uniform triangulation $\mathscr{E}_{\mathbf{h},A}$ defined above the corresponding classes of polynomials are still contained in the resulting nonuniform spline spaces. This property of $\mathscr{S}_{k,\mathbf{h}}(A,\Omega)$ should allow us to adapt if necessary the network of knots so that the domain on which the B-splines form a partition of unity is adjusted well to the given domain Ω. Of course, in the case that Ω is, say, the union of a finite number of s-rectangles this can be achieved without passing to irregular configurations.

As a model we want to construct now a scale of configurations of knot sets which will be adjusted in the above sense for simplicity to $\Omega=[0,1]^s$. Given a matrix A satisfying (4.2) and (4.5) we assign to any $m\in N$ a mesh size $\mathbf{h}$ by setting

$$\mathbf{h}_1 := (\mathbf{1}+\mathbf{a}^+ +\mathbf{a}^-), \qquad \mathbf{h} := \mathbf{h}_1/((m-1)\,\mathbf{h}_1+\mathbf{1}), \tag{4.11}$$

where $\mathbf{a}^\pm$ are defined by (4.4). For any $m\in N$ the corresponding configurations of knot sets will be induced by the matrices

$$H := (\operatorname{diag}_s((m-1)\,\mathbf{h}_1+\mathbf{1}))^{-1} A \operatorname{diag}_n(\hat{\mathbf{h}}_1). \tag{4.12}$$

Defining the vectors $\mathbf{d}^\pm\in R^s_+$ analogously to (4.4) now with respect to H we consider similarly as before the affine maps $H_\mathbf{v}: R^n\to R^n$ defined by

$$H_\mathbf{v}(\mathbf{u}) := H\mathbf{u}+\hat{\mathbf{h}}\mathbf{v}-\mathbf{d}^+, \qquad \mathbf{v}\in Z^s_t, \quad \mathbf{v}\leqslant m\cdot\mathbf{1}, \tag{4.13}$$

and the corresponding configurations of knot sets

$$P_{\pi,\mathbf{v}}=\{H_\mathbf{v}(\mathbf{v}_0^\pi)|_{R^s},\ldots,H_\mathbf{v}(\mathbf{v}_n^\pi)|_{R^s}\}, \qquad \mathscr{P}_H=\{P_{\pi,\mathbf{v}}:\mathbf{v}\leqslant m\cdot\mathbf{1},\pi\in\operatorname{Per}_n\}. \tag{4.14}$$

We wish to discuss the corresponding spline spaces being again of type (3.2).

$$\mathscr{S}_k(H) := \operatorname{span}\{M(x\mid P): P\in\mathscr{P}_H\}. \tag{4.15}$$

For later reference we list some elementary properties of this setting.

LEMMA 4.2. *Let h and H be defined by* (4.11) *and* (4.12), *respectively. Then one has*

(i) $d^\pm=\mathbf{a}^\pm/((m-1)\,\mathbf{h}_1+\mathbf{1})$, $\mathbf{d}^\pm/\mathbf{h}=\mathbf{a}^\pm/\mathbf{h}_1$, $m\mathbf{h}=\mathbf{1}+\mathbf{d}^++\mathbf{d}^-$;

(ii) $\{\mathbf{x}\in R^s:\operatorname{vol}_k(\{\mathbf{u}\in H(Q):\mathbf{u}|_{R^s}=\mathbf{x}\})=1\}=[\mathbf{d}^+,\mathbf{h}-\mathbf{d}^-]$;

(iii) *Ω is exactly the domain where the B-splines arising from $\mathscr{P}_H$ form a partition of unity, i.e.,*

$$\Omega=\left\{\mathbf{x}\in R^s:\operatorname{vol}_k\left(\{\mathbf{u}\in\bigcup\{H_\mathbf{v}(Q):\mathbf{v}\leqslant m\cdot\mathbf{1}\}:\mathbf{u}|_{R^s}=\mathbf{x}\}\right)=1\right\}.$$

(iv) *For every* $\mathbf{v} \leqslant m \cdot \mathbf{1}$ *and* $\pi \in \mathrm{Per}_n$ *one has*

$$\mathrm{vol}_s([P_{\pi,\mathbf{v}}])/\mathrm{vol}_s(\Omega \cap [P_{\pi,\mathbf{v}}]) \leqslant s!((1+2b)/(1-2b))^s,$$

where b *is the constant appearing in* (4.5).

(v) *The global smoothness of the splines in* $\mathscr{S}_k(H)$ *is for sufficiently small* $\mathbf{h}$ *completely determined by the matrix* A, *specifically*, H *is dispersive for sufficiently small* $\mathbf{h}$ *iff* $A_0 := A\ \mathrm{diag}_n(\hat{\mathbf{h}}_1)$ *is dispersive.*

Proof. (i) and (ii) are immediate consequences of the definitions of $\mathbf{d}^{\pm}$ and (4.11). (ii) and (4.13) provide (iii). From (ii) and (iii) and (4.14) we conclude now that

$$\mathrm{vol}_s([P_{\pi,\mathbf{v}}] \cap \Omega) \geqslant (1/s!)\,\mathrm{vol}_s([\mathbf{d}^+, \mathbf{h} - \mathbf{d}^-]) \geqslant (1/s!)(1-2b)^s \cdot \mathbf{h}^{\mathbf{1}}$$

holds for any $\pi \in \mathrm{Per}_n$, $\mathbf{v} \leqslant m \cdot \mathbf{1}$. Thus (iv) follows from the estimate

$$\mathrm{vol}_s([P_{\pi,\mathbf{v}}]) \leqslant \mathrm{vol}_s(H_{\mathbf{v}}(Q)|_{R^s}) \leqslant (1+2b)^s \cdot \mathbf{h}^{\mathbf{1}}$$

Setting $mH := (u_{ij})_{i,j=1}^n$ and $A_0 = A\ \mathrm{diag}_n(\hat{\mathbf{h}}_1) := (v_{ij})_{i,j=1}^n$, (4.11) and (4.12) yield

$$\lim_{m\to\infty} u_{ij} = v_{ij}, \qquad i = 1,\ldots, s, \quad j = 1,\ldots, n.$$

This completes the proof.

Let us close this section with some brief remarks concerning the dimensionality of $\mathscr{S}_k(H)$. Setting

$$\mathscr{E}_H = \bigcup \{\mathscr{K}_n(H_{\mathbf{v}}(Q)) : \mathbf{v} \leqslant m \cdot \mathbf{1}\} \tag{4.16}$$

we have the estimates

$$m^s \leqslant \dim \mathscr{S}_k(H) \leqslant \mathrm{card}(\mathscr{P}_H) \leqslant \mathrm{card}(\mathscr{E}_H) = n!m^s.$$

However, we shall see below (cf. the remark subsequent to (5.5)) that when $k > 1$ the upper bounds are not sharp, i.e., $\dim \mathscr{S}_k(H) < \mathrm{card}(\mathscr{P}_H)$. A more detailed discussion concerning the precise dimension of such spaces will follow elsewhere. Here we mention only that one has for all choices of A, $\dim \mathscr{S}_k(H) = \mathrm{card}(\mathscr{E}_H)$ iff $k = 0, 1$.

5. Local Approximation Schemes

In this section we construct and discuss certain linear local approximation processes which will be related to the uniform configurations given by (4.14). More precisely, we are interested in operators of the following type:

$$Q(H \mid f)(x) := \sum_{\pi, \mathbf{v}} \lambda_{\pi,\mathbf{v}}(f)\, N(\mathbf{x} \mid \pi, \mathbf{v}), \tag{5.1}$$

where according to (4.14) $\sum_{\pi,\mathbf{v}}$ will always mean summation over $\pi \in \mathrm{Per}_n$ and $\mathbf{v} \in Z_+^s$, $\mathbf{v} \leqslant m \cdot \mathbf{1}$. It is somewhat more convenient to use here the "normalized" B-splines (cf. (4.11), (4.14))

$$N(\mathbf{x} \mid \pi, \mathbf{v}) := (\mathbf{h}^{\mathbf{1}}/n!)\, M(\mathbf{x} \mid P_{\pi,\mathbf{v}}). \tag{5.2}$$

Indeed since $\mathbf{h}^{\mathbf{1}}/n! = \mathrm{vol}_n(\sigma)$ for all $\sigma \in \mathscr{E}_H$ (cf. (4.16)), we have (cf. (2.8), (2.9), (3.1), Lemma 4.2(iii))

$$\sum_{\pi,\mathbf{v}} N(\mathbf{x} \mid \pi, \mathbf{v}) = 1 \qquad \text{for all} \quad \mathbf{x} \in \Omega. \tag{5.3}$$

Following the ideas in [5, 18], $\lambda_{\pi,\mathbf{v}}(f)$ will be linear functionals which are to be chosen in such a way that $Q(H, \cdot)$ reproduces all polynomials up to the degree of the splines. The construction of these functionals is based on representations of the monomials $\mathbf{x}^\alpha$ as linear combinations of the B-splines $N(\mathbf{x} \mid \pi, \mathbf{v})$.

To this end, we consider first for $\mathbf{w} = (\mathbf{1} + 2(\mathbf{a}^+ + \mathbf{a}^-))^{-1}$ the affine transformations

$$\begin{aligned} L(\mathbf{x}) &= \mathbf{w}(\mathbf{x} + \mathbf{a}^+ + \mathbf{a}^-), \qquad && \mathbf{x} \in \Omega, \\ \hat{L}(\mathbf{u}) &= \hat{\mathbf{w}}(\mathbf{u} + \mathbf{a}^+ + \mathbf{a}^-), \qquad && \mathbf{u} \in \Omega \times [0, 1]^k. \end{aligned} \tag{5.4}$$

Setting for a given $\pi \in \mathrm{Per}_n$ and $\mathbf{v} \in Z_+^s$, $\mathbf{u}_j = \hat{L}(A_0(\mathbf{v}_j^\pi) - \mathbf{a}^+ + \mathbf{v}\mathbf{h}_1)$, $j = 0, ..., n$, we define for any $\boldsymbol{\alpha} \in Z_+^s$, $|\boldsymbol{\alpha}| \leqslant k$, and any index set I, $|I| = |\boldsymbol{\alpha}|$ as in (3.5).

$$\eta(I, \pi, \mathbf{v}) := \det \begin{pmatrix} F_1(\mathbf{u}_0; I) & \cdots & F_1(\mathbf{u}_n; I) \\ 1 & \cdots & 1 \end{pmatrix}. \tag{5.5}$$

It is readily seen that to any $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| \leqslant k$, we can assign $r(\boldsymbol{\alpha}) = \binom{k}{\alpha_1} \cdot \binom{k-\alpha_1}{\alpha_2} \cdot \binom{k-\alpha_1-\cdots-\alpha_{s-1}}{\alpha_s} = \binom{k}{\boldsymbol{\alpha}}$ different mappings $F_1(\cdot, I)$, $|I| = \boldsymbol{\alpha}$. So, we consider the averages

$$\eta^{\boldsymbol{\alpha}}(\pi, \mathbf{v}) := \Big(\sum_{|I| = \boldsymbol{\alpha}} \eta^{\boldsymbol{\alpha}}(I, \pi, \mathbf{v}) \Big) \Big/ r(\boldsymbol{\alpha}). \tag{5.6}$$

The following relation between $\eta^{\alpha}(\pi, \mathbf{v})$ and $\eta^{\alpha}(\pi, \mathbf{0})$, $\mathbf{v} \in Z^s_+$, will play a crucial role.

LEMMA 5.1. *Let* $\eta^{\alpha}(\pi, \mathbf{v})$ *defined by* (5.5) *and* (5.6); *then the following relation holds*

$$\eta^{\alpha}(\pi, \mathbf{v}) = \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} (\mathbf{w}\mathbf{h}_1 \mathbf{v})^{\boldsymbol{\beta}} \eta^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\pi, \mathbf{0}).$$

Proof. Writing for any $|I| = |\boldsymbol{\alpha}|$

$$\begin{aligned}\eta(I, \pi, \mathbf{v}) &= (-1)^n \det(F_1(\mathbf{u}_1; I) - F_1(\mathbf{u}_0; I) \cdots F_1(\mathbf{u}_n; I) - F_1(\mathbf{u}_0; I)) \\ &=: (-1)^n \det(A(I, \pi, \mathbf{v}))\end{aligned}$$

it follows immediately from (3.5), (4.13) and the definition of $\mathbf{u}_j = \mathbf{u}_j(\pi, \mathbf{v})$ that the matrix $A(I, \pi, \mathbf{v})$ takes the following form

$$A(I, \pi, \mathbf{v}) = \begin{pmatrix} (\mathbf{y}_1)_1 & \cdots & (\mathbf{y}_n)_1 \\ \vdots & & \vdots \\ (\mathbf{y}_1)_s & \cdots & (\mathbf{y}_n)_s \\ \vdots & & \vdots \\ \varepsilon_{j1}(\hat{\mathbf{y}}_1 + \mathbf{h}_1 \mathbf{w}\mathbf{v})_j & \cdots & \varepsilon_{jn}(\hat{\mathbf{y}}_n + \mathbf{h}_1 \mathbf{w}\mathbf{v})_j \\ \vdots & & \vdots \end{pmatrix}$$

with $\varepsilon_{ji} \in \{0, 1\}$depending on π and $\mathbf{y}_j = \mathbf{w}A_0 \mathbf{v}_j^{\pi}$, $\hat{\mathbf{y}}_j = \mathbf{y}_j + \mathbf{a}^{-}\mathbf{w}$. Since the determinant is linear with respect to the rows we obtain

$$\det(A((I, \pi, \mathbf{v})) = \det(B_1) + (\mathbf{h}_1 \mathbf{w}\mathbf{v})_j \det(B_2),$$

where the matrices B_i, $i = 1, 2$, coincide with $A(I, \pi, \mathbf{v})$ in all but the jth row and $(\varepsilon_{j1}(\hat{\mathbf{y}}_1)_j \cdots \varepsilon_{jn}(\hat{\mathbf{y}}_n)_j)$ and $(\varepsilon_{j1}, \dots, \varepsilon_{jn})$ are the jth rows of B_1 and B_2, respectively. Applying this decomposition successively to all the rows indicated by I, and observing that for any $\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}$ there are $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$ distinct subsets I', $|I'| = |\boldsymbol{\beta}|$, for which

$$F_1(F_1(\mathbf{u}; I \backslash I'); I') = F_1(\mathbf{u}; I)$$

we obtain

$$\eta(I, \pi, \mathbf{v}) = \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} (\mathbf{v}\mathbf{w}\boldsymbol{h}_1)^{\boldsymbol{\beta}} \Theta^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(I, \pi, \mathbf{0}),$$

where

$$\Theta^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(1, \pi, \mathbf{0}) := \left(\sum_{j=1}^{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}} \eta(I \backslash I'; \pi, \mathbf{0})\right) \Big/ \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}}.$$

Moreover, it is not hard to realize that

$$\left(\sum_{|I|=|\alpha|} \Theta^{\alpha-\beta}(I,\pi,\mathbf{0})\right)\Big/ r(\alpha) = \eta^{\alpha-\beta}(\pi,\mathbf{0})$$

which in view of (5.6) completes the proof.

Defining now

$$\mu_\pi^\alpha = \mathbf{w}^{-\alpha} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-\mathbf{w}(\mathbf{a}^+ + \mathbf{a}^-))^{\alpha-\beta} \eta^\beta(\pi,\mathbf{0}) \tag{5.7}$$

and for any $\mathbf{h}$ given by (4.11)

$$\mu_{\pi,\mathbf{v}}^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (\mathbf{h}_1 \mathbf{v})^\beta \mu_\pi^{\alpha-\beta}, \qquad \xi_{\pi,\mathbf{v}}^\alpha = (\mathbf{h}/\mathbf{h}_1)^\alpha \mu_{\pi,\mathbf{v}}^\alpha \tag{5.8}$$

we arrive at the following lemma.

LEMMA 5.2. *Using the above notation one has for* $|\alpha| \leqslant k$

$$\sum_{\pi,\mathbf{v}} \xi_{\pi,\mathbf{v}}^\alpha N(\mathbf{x} \mid \pi, \mathbf{v}) = \mathbf{x}^\alpha \qquad \textit{for} \quad \mathbf{x} \in \Omega. \tag{5.9}$$

Proof. Let us define similarly as in (5.4) for $\mathbf{v} = (\mathbf{1} + 2(\mathbf{d}^+ + \mathbf{d}^-))^{-1}$ the transformations

$$\begin{aligned} Y(\mathbf{x}) &= \mathbf{v}(\mathbf{x} + \mathbf{d}^+ + \mathbf{d}^-), && \mathbf{x} \in \Omega,\\ \hat{Y}(\hat{\mathbf{u}}) &= \hat{\mathbf{v}}(\mathbf{u} + \mathbf{d}^+ + \mathbf{d}^-), && \mathbf{u} \in \Omega \times [0,1]^k. \end{aligned} \tag{5.10}$$

Defining furthermore for $\pi \in \mathrm{Per}_n$, $\mathbf{v} \leqslant m \cdot \mathbf{1}$, $|\alpha| \leqslant k$, $\Phi(I,\pi,\mathbf{v})$, $\Phi^\alpha(\pi,\mathbf{v})$ as in (5.5) and (5.6) but with $\mathbf{u}_j$ replaced by the vectors $\hat{Y}(H_\mathbf{v}(\mathbf{v}_j^\pi))$ Lemma 4.2(i) yields

$$\Phi^\alpha(\pi,\mathbf{v}) = (\mathbf{v}\mathbf{h}/\mathbf{w}\mathbf{h}_1)^\alpha \eta^\alpha(\pi,\mathbf{v}). \tag{5.11}$$

Moreover, we observe that Y takes $[-\mathbf{d}^+ - \mathbf{d}^-, \mathbf{1} + \mathbf{d}^+ + \mathbf{d}^-]$ into Ω and

$$\tilde{\mathscr{E}} = \{\hat{Y}(\sigma) : \sigma \in \mathscr{E}_H\}$$

is a triangulation of $\Omega_1 \times [0,1]^k$, where $\Omega_1 = [\mathbf{v}(\mathbf{d}^+ + \mathbf{d}^-), \mathbf{v}(\mathbf{1} + \mathbf{d}^+ + \mathbf{d}^-)] \subset \Omega$. Note that all the vertices of simplices in $\tilde{\mathscr{E}}$ belong to $\bigcup \{\Omega \times \{\mathbf{u}\} : \mathbf{u}$ is a vertex of $[0,1]^k\}$. It is then not hard to see that the arguments of [11, Theorem 2.1] hold in this particular case for $t = 1$, where t

occurs in the map $F_l(\cdot\,; I)$ (cf. (3.5)). Hence (3.7) implies on account of the definition of $\Phi^\alpha(\pi, \mathbf{v})$

$$\mathbf{z}^\alpha = \sum_{\pi,\mathbf{v}} \Phi^\alpha(\pi, \mathbf{v})\, \tilde{N}(\mathbf{z} \mid \pi, \mathbf{v}), \tag{5.12}$$

where $\tilde{N}(\mathbf{z} \mid \pi, \mathbf{v})$ are the "normalized" B-splines with respect to the configuration of knot sets induced by $\tilde{\mathscr{E}}$, i.e., when $\mathbf{z} = Y(\mathbf{x}) \in \Omega$,

$$\tilde{N}(\mathbf{z} \mid \pi, \mathbf{v}) = N(\mathbf{x} \mid \pi, \mathbf{v}). \tag{5.13}$$

In view of the definition (5.8) we have now

$$\xi^\alpha_{\pi,\mathbf{v}} = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (\mathbf{h}_1 \mathbf{v})^\beta \left[\mathbf{w}^{\beta-\alpha} \sum_{\gamma \leqslant \alpha-\beta} \binom{\alpha-\beta}{\gamma} (-\mathbf{w}(\mathbf{a}^+ + \mathbf{a}^-))^{\alpha-\beta-\gamma} \eta^\gamma(\pi, \mathbf{0}) \right].$$

Since $\binom{\alpha}{\beta}\binom{\alpha-\beta}{\gamma} = \binom{\alpha}{\gamma}\binom{\alpha-\gamma}{\beta}$ we can rewrite the right hand side of the above equation after reordering the summands as

$$\sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \mathbf{w}^{-\alpha} \left(\sum_{\beta \leqslant \alpha-\gamma} \binom{\alpha-\gamma}{\beta} (\mathbf{h}_1 \mathbf{w}\mathbf{v})^\beta \eta^{\alpha-\gamma-\beta}(\pi, \mathbf{0})(-\mathbf{w}(\mathbf{a}^+ + \mathbf{a}^-))^\gamma \right).$$

By virtue of Lemma 5.1 and (5.11) we obtain

$$\xi^\alpha_{\pi,\mathbf{v}} = \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \mathbf{w}^{-\alpha}(-\mathbf{w}(\mathbf{a}^+ + \mathbf{a}^-))^\gamma (\mathbf{w}\mathbf{h}_1/\mathbf{v}\mathbf{h})^{\alpha-\gamma}\, \Phi^{\alpha-\gamma}(\pi, \mathbf{v})$$

which by (4.11), Lemma 4.2(i) and the definitions of $\mathbf{v}$, $\mathbf{w}$ reduces to

$$\xi^\alpha_{\pi,\mathbf{v}} = \mathbf{v}^{-\alpha} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-\mathbf{v}(\mathbf{d}^+ + \mathbf{d}^-))^\gamma\, \Phi^{\alpha-\gamma}(\pi, \mathbf{v}).$$

Thus, we get by (5.10), (5.12) and (5.13)

$$\begin{aligned}
\sum_{\pi,\mathbf{v}} \xi^\alpha_{\pi,\mathbf{v}} N(\mathbf{x} \mid \pi, \mathbf{v}) &= \sum_{\pi,\mathbf{v}} \mathbf{v}^{-\alpha} \left(\sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-\mathbf{v}(\mathbf{d}^+ + \mathbf{d}^-))^\gamma \right. \\
&\qquad \left. \times\, \Phi^{\alpha-\gamma}(\pi, \mathbf{v})\, \tilde{N}(\mathbf{z} \mid \pi, \mathbf{v}) \right) \\
&= \mathbf{v}^{-\alpha} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-\mathbf{v}(\mathbf{d}^- + \mathbf{d}^-))^\gamma \mathbf{z}^{\alpha-\gamma} \\
&= \mathbf{v}^{-\alpha}(\mathbf{z} - \mathbf{v}\,(\mathbf{d}^+ + \mathbf{d}^-))^\alpha = (Y^{-1}(\mathbf{z}))^\alpha = \mathbf{x}^\alpha
\end{aligned}$$

which completes the proof.

Note that the coefficients $\xi^{\alpha}_{\pi,\mathbf{v}}$ are essentially averages over volumes of simplices (cf. (3.7)) which are obtained by different choices of the index sets I, $|I| = |\boldsymbol{\alpha}|$. So, for $k > 1$ the $\xi^{\alpha}_{\pi,\mathbf{v}}$ do not form a unique collection of coefficients satisfying (5.9). This means, in accordance with the remarks at the end of Section 4, that the $N(\mathbf{x} \mid \pi, \mathbf{v})$ are in general not linearly independent. Since we merely want to show in principle that in spite of this lack the ideas in [5, 18] can be made to work in the present setting, by using the special construction (5.8), we restrict ourselves to the discussion of a possibly simple type of operators $Q(H; \cdot)$ which involve evaluations of derivatives of the approximated function f. Further variants which require only point evaluations of f itself are discussed in [13].

Fixing $\boldsymbol{\delta}_\pi$ in $[\{A_0\mathbf{v}^\pi_0 - \mathbf{a}^+, ..., A_0\mathbf{v}^\pi_n - \mathbf{a}^+\}] \cap \Omega$ we set for $\mathbf{h}$ given by (4.11)

$$\boldsymbol{\tau}_{\pi,\mathbf{v}} = \mathbf{h}\mathbf{v} + \boldsymbol{\delta}_\pi/((m-1)\,\mathbf{h}_1 + \mathbf{1}) \tag{5.14}$$

and define for $|\boldsymbol{\alpha}| \leqslant k$ and $g \in C^k(R^s)$ the linear functionals

$$\lambda^{\alpha}_{\pi,\mathbf{v}}(g) = (D^{\alpha}g)(\boldsymbol{\tau}_{\pi,\mathbf{v}}). \tag{5.15}$$

Ordering the multi-indices lexicographically it is readily seen that

$$(\lambda^{\alpha}_{\pi,\mathbf{v}}(\mathbf{x}^{\beta}))_{|\alpha|,|\beta| \leqslant k}$$

is a triangular matrix with the constant diagonal entries $\boldsymbol{\alpha}!$ Hence the system

$$\sum_{\alpha \leqslant \beta} a^{\alpha}_{\pi,\mathbf{v}} \lambda^{\alpha}_{\pi,\mathbf{v}}(\mathbf{x}^{\beta}) = \xi^{\beta}_{\pi,\mathbf{v}}, \qquad |\boldsymbol{\beta}| \leqslant k, \tag{5.16}$$

has a unique solution $(a^{\alpha}_{\pi,\mathbf{v}})_{|\alpha| \leqslant k}$. Setting now

$$\lambda_{\pi,\mathbf{v}} = \sum_{|\alpha| \leqslant k} a^{\alpha}_{\pi,\mathbf{v}} \lambda^{\alpha}_{\pi,\mathbf{v}} \tag{5.17}$$

one has $\lambda_{\pi,\mathbf{v}}(\mathbf{x}^{\alpha}) = \xi^{\alpha}_{\pi,\mathbf{v}}$ so that for $g \in \Pi_k$

$$Q(H, g) = \sum_{\pi,\mathbf{v}} \lambda_{\pi,\mathbf{v}}(g)\, N(\mathbf{x} \mid \pi, \mathbf{v}) = g(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{5.18}$$

Moreover, the polynomials $g^{\alpha}_{\pi,\mathbf{v}}(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\tau}_{\pi,\mathbf{v}})^{\alpha}/\boldsymbol{\alpha}! \in \Pi_k$ satisfy

$$\lambda^{\alpha}_{\pi,\mathbf{v}}(g^{\beta}_{\pi,\mathbf{v}}) = \delta_{\alpha,\beta}. \tag{5.19}$$

Observing (5.16), (5.17) and (5.19) when applying $\lambda_{\pi,\mathbf{v}}$ to both sides of the equality

$$(\mathbf{x} - \boldsymbol{\tau}_{\pi,\mathbf{v}})^{\alpha} = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \mathbf{x}^{\beta}(-\boldsymbol{\tau}_{\pi,\mathbf{v}})^{\alpha-\beta}$$

we obtain

$$a^{\alpha}_{\pi,v} = (1/\alpha!) \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-\tau_{\pi,v})^{\alpha-\beta} \xi^{\beta}_{\pi,v}. \tag{5.20}$$

This yields in view of the homogeneity of $\xi^{\beta}_{\pi,v}$ (cf. (5.8))

$$a^{\alpha}_{\pi,v} = (\mathbf{h}/\mathbf{h}_1)^{\alpha} b^{\alpha}_{\pi,v}, \tag{5.21}$$

where $b^{\alpha}_{\pi,v}$ satisfies for $\mathbf{v} \in Z^s_+$

$$\sum_{|\alpha| \leqslant k} b^{\alpha}_{\pi,v} D^{\alpha}(\mathbf{x}^{\beta})|_{\delta_\pi + \mathbf{v}\mathbf{h}_1} = \mu^{\beta}_{\pi,v}, \qquad |\alpha|, |\beta| \leqslant k. \tag{5.22}$$

In particular, the special structure of the $\xi^{\alpha}_{\pi,v}$ provides

LEMMA 5.3. *Let $a^{\alpha}_{\pi,v}$ be defined by* (5.20) *then one has*

$$a^{\alpha}_{\pi,v} = a^{\alpha}_{\pi,0}, \qquad \mathbf{v} \in Z^s_+. \tag{5.23}$$

Proof. Considering

$$\begin{aligned}\sum_{\alpha<\beta} a^{\alpha}_{\pi,0} \lambda^{\alpha}_{\pi,v}(\mathbf{x}^{\beta}) &= \sum_{\alpha<\beta} a^{\alpha}_{\pi,0} \lambda^{\alpha}_{\pi,0}((\mathbf{x}+\mathbf{h}\mathbf{v})^{\beta}) \\ &= \sum_{\gamma<\beta} \binom{\beta}{\gamma} (\mathbf{h}\mathbf{v})^{\gamma} \left(\sum_{\alpha<\beta-\gamma} \left(\sum_{\alpha<\beta-\gamma} a^{\alpha}_{\pi,0} \lambda^{\alpha}_{\pi,0}(\mathbf{x}^{\beta-\gamma}) \right)\right. \end{aligned} \tag{5.24}$$

and using (5.16) with $\mathbf{v} = \mathbf{0}$ the right hand side of (5.24) simplifies to

$$\sum_{\gamma<\beta} \binom{\beta}{\gamma} (\mathbf{h}\mathbf{v})^{\gamma} \xi^{\beta-\gamma}_{\pi,0}$$

which in view of (5.8) is equal to $\xi^{\beta}_{\pi,v}$. The assertion follows now from the uniqueness of the solution of (5.16).

Clearly (5.23) implies by (5.21) that also

$$b^{\alpha}_{\pi,v} = b^{\alpha}_{\pi,0}, \qquad \mathbf{v} \in Z^s_+,$$

holds which means that for $|\alpha| \leqslant k$, $\mathbf{v} \in Z^s_+$, $\pi \in \mathrm{Per}_n$,

$$|a^{\alpha}_{\pi,v}| \leqslant (\max\{|b^{\alpha}_{\pi,0}| : |\alpha| \leqslant k, \pi \in \mathrm{Per}_n\}/\mathbf{h}^{\alpha}_1)\, \mathbf{h}^{\alpha}. \tag{5.25}$$

This leads to (cf. [9])

LEMMA 5.4. *For each* $\pi \in \mathrm{Per}_n$ *and* $\mathbf{v} \leqslant m \cdot \mathbf{1}$ (*cf.* (4.11)) *there is a functional* $\tilde{\lambda}_{\pi,\mathbf{v}} \in L_p^*(\Omega)$ *supported in* $[P_{\pi,\mathbf{v}}] \cap \Omega$ *such that*

(i) $\tilde{\lambda}_{\pi,\mathbf{v}}(g) = \lambda_{\pi,\mathbf{v}}(g)$ *for all* $g \in \Pi_k(\Omega)$, *where* $\lambda_{\pi,\mathbf{v}}$ *is defined by* (5.17).

(ii) $|\tilde{\lambda}_{\pi,\mathbf{v}}(f)| \leqslant C \operatorname{vol}_s([P_{\pi,\mathbf{v}}])^{-1/p} \|f\|_p([P_{\pi,\mathbf{v}}] \cap \Omega)$, *where* C *depends only on* s, k *and* b (*cf.* (4.5)).

Proof. By the definition (5.17) we have for $g \in \Pi_k(\Omega)$

$$|\lambda_{\mathbf{v}}(g)| \leqslant \sum_{|\alpha| \leqslant k} |a^{\alpha}_{\pi,\mathbf{v}}| \, |\lambda^{\alpha}_{\pi,\mathbf{v}}(g)|.$$

Setting $h = \max\{h_i : i = 1,..., s\}$, (5.25) provides

$$|a^{\alpha}_{\pi,\mathbf{v}}| \, |\lambda^{\alpha}_{\pi,\mathbf{v}}(g)| \leqslant C h^{|\alpha|} \, |D^{\alpha} g(\boldsymbol{\tau}_{\pi,\mathbf{v}})|. \tag{5.26}$$

By Lemma 4.2(iv) we can find a cube $I_{\pi,\mathbf{v}} \subset [P_{\pi,\mathbf{v}}] \cap \Omega$ such that

$$\operatorname{vol}_s(I_{\pi,\mathbf{v}}) \geqslant C \operatorname{vol}_s([P_{\pi,\mathbf{v}}]),$$

where $C > 0$ depends only on s and b (cf. (4.5)). Assuming that $\boldsymbol{\tau}_{\pi,\mathbf{v}} \in I_{\pi,\mathbf{v}}$ a standard scaling argument provides

$$|\lambda_{\pi,\mathbf{v}}(g)| \leqslant C \operatorname{vol}_s([P_{\pi,\mathbf{v}}])^{-1/p} \, \|g\|_p \, ([P_{\pi,\mathbf{v}}] \cap \Omega),$$

where C depends only on s, k and b. Now, the Hahn–Banach theorem guarantees the existence of a norm preserving extension $\tilde{\lambda}_{\pi,\mathbf{v}}$ of $\lambda_{\pi,\mathbf{v}}$ from $\Pi_k([P_{\pi,\mathbf{v}}])$ to all of $L_p([P_{\pi,\mathbf{v}}])$.

In the following, we shall briefly write $\lambda_{\pi,\mathbf{v}}$ instead of $\tilde{\lambda}_{\pi,\mathbf{v}}$.

We are now in a position to estimate the remainder $\|D^{\alpha}(f - Q(H, f))\|_p(\Omega)$ by following standard lines (cf. [5, 18]) where $Q(H, \cdot)$ is given by (5.18) with respect to the functionals in Lemma 5.4. By Lemma 5.4(i) and (5.18) we have whenever $D^{\alpha} f$ and $D^{\alpha} Q(H, f)$ exist

$$\begin{aligned} D^{\alpha}(f - Q(H, f))(\mathbf{x}) &= D^{\alpha}(f - g)(\mathbf{x}) + D^{\alpha}(Q(H, f - g))(\mathbf{x}) \\ &= D^{\alpha}(f - g)(\mathbf{x}) + \sum_{(\pi, \mathbf{v}) \in \Psi(\mathbf{x})} \lambda_{\pi,\mathbf{v}}(g - f) \, D^{\alpha} N(\mathbf{x} \mid \pi, \mathbf{v}), \end{aligned}$$

where $\Psi(\mathbf{x}) = \{(\pi, \mathbf{v}) : \mathbf{x} \in [P_{\pi,\mathbf{v}}]\}$ and g is any polynomial in $\Pi_k(\Omega)$. Denoting by $Q_{\pi,\mathbf{v}}$ the smallest cube containing $[P_{\pi,\mathbf{v}}] \cap \Omega$ and setting $\Xi_{\pi',\mathbf{v}'} = \bigcup \{\Psi(\mathbf{x}) : \mathbf{x} \in Q_{\pi',\mathbf{v}'}\}$ one gets by virtue of Lemma 5.4(ii)

$$\begin{aligned} \|D^{\alpha}(f - Q(H, f))\|_p(Q_{\pi',\mathbf{v}'}) &\leqslant \|D^{\alpha}(f - g)\|_p(Q_{\pi',\mathbf{v}'}) \\ &\quad + C \sum_{(\pi, \mathbf{v}) \in \Xi_{\pi',\mathbf{v}'}} \|f - g\|_p(Q_{\pi,v}) \operatorname{vol}_s([P_{\pi,\mathbf{v}}])^{-1/p} \\ &\quad \times \left(\int_{[P_{\pi,\mathbf{v}}]} |D^{\alpha} N(\mathbf{x} \mid \pi, \mathbf{v})|^p \, d\mathbf{x} \right)^{1/p}. \end{aligned} \tag{5.27}$$

A repeated application of (2.5) and (2.11) affirms in view of (4.13) and (4.14)

$$\|D^{\alpha}N(\cdot \mid \pi,\mathbf{v})\|_{\infty} = \|D^{\alpha}N(\cdot \mid \pi, \mathbf{0})\|_{\infty} \leqslant Ch^{-|\alpha|}. \tag{2.28}$$

When $\alpha = \mathbf{0}$ one clearly has $C = 1$. But when $|\alpha| > 0$, C will depend on the matrix H. More precisely, one may derive from (2.11) that C is of moderate size if at least $s+1+|\alpha|$ of the knots in $P_{\pi,0}$, $\pi \in \mathrm{Per}_n$ are sufficiently uniformly distributed (cf. the remarks following Theorem 4.1).

However Lemma 4.2(v) ensures us that the constant C is determined by the matrix $A_0 = H\,\mathrm{diag}_n(\hat{\mathbf{h}}_1)$ and hence does not depend on $\mathbf{h}$.

So, (5.27) and (5.28) yield with $\tilde{Q}_{\pi',v'} = \bigcup \{Q_{\pi,v} : (\pi, v) \in \Xi_{\pi',v'}\}$

$$\|D^{\alpha}(f - Q(H,f))\|_p(Q_{\pi',\mathbf{v}'}) \leqslant C\,\|D^{\alpha}(f-g)\|_p(Q_{\pi',\mathbf{v}'}) \\ + C\,\|f-g\|_p(\tilde{Q}_{\pi',\mathbf{v}'})\,h^{-|\alpha|}.$$

Choosing $g \in \Pi_k$ so that (cf. [9; 21, p. 85])

$$\|D^{\alpha}(f-g)\|_p(Q_{\pi',\mathbf{v}'}) \leqslant C(\mathrm{diam}\;Q_{\pi',\mathbf{v}'})^{k+1-|\alpha|}\,|f|_{p,k+1}(Q_{\pi',\mathbf{v}'}),$$

where $|f|_{p,k+1}(\Omega) = (\sum_{|\alpha|=k+1} \|D^{\alpha}f\|_p^p(\Omega))^{1/p}$, we obtain

$$\|D^{\alpha}(f-g)\|_p(Q_{\pi',\mathbf{v}'}) \leqslant Ch^{k+1-|\alpha|}\,|f|_{p,k+1}(\tilde{Q}_{\pi',\mathbf{v}'})$$

with C depending only on b, s and k. Hence

$$\|D^{\alpha}(f - Q(H,f))\|_p(Q_{\pi',\mathbf{v}'}) \leqslant Ch^{k+1-|\alpha|}\,|f|_{p,k+1}(\tilde{Q}_{\pi',\mathbf{v}'}). \tag{5.29}$$

So, after summing up the local estimates with respect to π', $\mathbf{v}'$ we arrive at the following result.

THEOREM 5.1. *Let $Q(H;\cdot)$ be defined by* (5.18) *with respect to the functionals $\lambda_{\pi,\mathbf{v}}$ given by Lemma* 5.4 *and suppose that $D^{\alpha}N(\mathbf{x} \mid \pi, \mathbf{v}) \in L_p(\Omega)$, $\pi \in \mathrm{Per}_n$, $\mathbf{v} \leqslant m \cdot 1$ (cf.* (4.11)) *and $f \in W_p^{k+1}(\Omega)$, then*

$$\|D^{\alpha}(f - Q(H:f))\|_p(\Omega) \leqslant Ch^{k+1-|\alpha|}\,|f|_{p,k+1}(\Omega),$$

where the constant C depends only on s, k and b (cf. (4.5)) *when $\alpha = \mathbf{0}$ and additionally on the matrix A_0 when $|\alpha| > 0$.*

Note that the approximation behavior for $|\alpha| > 0$ is essentially governed by the constant in (5.28). This motivates us to look for such dispersive matrices which are optimized in the sense of Section 4. Note also that the matrix A_0 plays the role of a local mesh ratio.

Using the equivalence of the K-functionals to corresponding moduli of smoothness (cf. [9, 15]) (at least for the domains considered here)

Theorem 5.1 immediately provides more general estimates for $f \in L_p(\Omega)$ in terms of $(k+1)$-th order moduli of smoothness.

Observing the relations (5.7), (5.8), (5.17), (5.21) and (5.23) we note that the construction of $Q(H;\cdot)$ requires essentially the computation of μ_π^α (cf. (5.7)) and $b_{\pi,0}^\pi$ (cf. (5.22)). So, when passing to a finer mesh size $\mathbf{h}$ the increase of computational work is caused only by the additional function evaluations. Recall that the essential relation (5.23) depends on the average structure of the $\xi_{\pi,\mathbf{v}}^\alpha$. In fact, numerical tests show that when replacing, e.g., for $k=3$, $s=2$, the coefficients $\eta^\beta(\pi,\mathbf{0})$ in (5.7) by $\eta^\beta(I,\pi,\mathbf{0})$ for some I, $|I|=|\beta|$, the errors increase for the same $\mathbf{h}$ by two digits while the computational work increases considerably, too.

Furthermore, one may reduce the number of function evaluations (by a factor $(s+k)!$) by setting for all $\pi \in \mathrm{Per}_n$

$$\boldsymbol{\tau}_{\pi,\mathbf{v}} = \boldsymbol{\tau}_\mathbf{v} = \boldsymbol{\tau}_0 + \mathbf{hv},$$

where, say $\boldsymbol{\tau}_0$ lies on the projection of the line connecting $\mathbf{0}$ and $\mathbf{1}_n$ in Q. Since this is an edge shared by all the simplices in $\mathscr{K}_n$ (cf. (4.1)) $\boldsymbol{\tau}_\mathbf{v}$ lies in (the closure of) all the supports $[P_{\pi,\mathbf{v}}]$ of the B-splines arising from $\mathscr{K}_n$.

Finally, one may replace the functionals $\lambda_{\pi,\mathbf{v}}^\alpha$ (cf. (5.20)) by appropriate difference quotients so that the application of the corresponding approximation schemes requires only the knowledge of function values rather than derivatives. This is pointed out in more detail in [13]. As expected, Theorem 5.1 holds for these schemes, too.

The above results have been confirmed by some numerical tests which were carried through by A. Krayer and P. Scharnagl. Table 5.1 shows as an example the L_∞-approximation errors obtained for different degrees k and the mesh sizes $\mathbf{h}=(\frac{1}{8},\frac{1}{8})$, $\mathbf{h}=(\frac{1}{16},\frac{1}{16})$ when approximating $f(x,y)=\sin(\pi(x+y))$ by a process of the above type. The last column in Table 5.1 illustrates the effect of replacing the averages $\eta^\alpha(\pi,\mathbf{0})$ by the volumes $\eta(I,\pi,\mathbf{v})$ for some fixed index set I (cf. (5.5), (5.6)).

TABLE 5.1

$\mathbf{h}$	k	Error	Error[a]
(0.125, 0.125)	0	0.0507472	0.0507472
(0.125, 0.125)	1	0.0129257	0.0129257
(0.125, 0.125)	2	0.0008099	0.0018260
(0.125, 0.125)	3	0.000037	0.0019437
(0.0625, 0.0625)	1	0.0024782	0.0024782
(0.0625, 0.0625)	2	0.0000722	0.0003460
(0.0625, 0.0625)	3	0.0000067	0.0009446

[a] From replacing the averages $\eta^\alpha(\pi,\mathbf{0})$ by the volumes $\eta(I,\pi,\mathbf{v})$ for some fixed index set I.

6. Local Refinements

Refining a uniform grid locally is known to be an efficient practical method to adapt on one hand the singularities, say, of the approximated function, whereas on the other hand many practical advantages of uniform grids are preserved. But in general, the construction of refinements which are to be compatible with a certain higher order global smoothness is known to be rather hard or sometimes even impossible.

In this section, we propose how to increase the flexibility of the above uniform spline spaces by generating "local refinements." The idea is to refine locally the "higher dimensional triangulation $\mathscr{E}_H$ (cf. (4.16)). In fact, decomposing the parallelepiped $H_\mathbf{v}(Q)$, $Q=[0,1]^n$ into the 2^n congruent parallelepipeds $H_\mathbf{v}(Q_\mu)$, where $Q_\mu = \operatorname{diag}_n(\frac{1}{2}\cdots\frac{1}{2})(Q)+\boldsymbol{\mu}$, $\mu_i \in \{0,\frac{1}{2}\}$, it is readily seen that

$$\mathscr{K}' = \bigcup \{\mathscr{K}_n(Q_\mu) : \mu_i \in \{0,\tfrac{1}{2}\}, i=1,\dots,n\}$$

is a triangulation of $H_\mathbf{v}(Q)$. Moreover, using Proposition 4.1(ii) one may easily show by induction that $\mathscr{K}'$ is actually a refinement of $\mathscr{K}_n(H_\mathbf{v}(Q))$, i.e.,

$$\text{Every element of } \mathscr{K}_n(H_\mathbf{v}(Q)) \text{ is the union of elements of } \mathscr{K}'. \tag{6.1}$$

Repeating this process one may keep on decomposing some or even all the $H_\mathbf{v}(Q_\mu)$, i.e., given $\mathscr{E}_H$ (cf. (4.16)), one may assign to every $\mathbf{v} \leqslant m\cdot\mathbf{1}$ a partition $\rho_\mathbf{v}$ of the above type so that the global refinement

$$\mathscr{R} = \{\rho_\mathbf{v} : \mathbf{v} \leqslant m\cdot\mathbf{1}\}$$

gives rise to the spline spaces

$$\mathscr{S}_k(H,\mathscr{R}) = \operatorname{span}\{M(\mathbf{x}\mid P(\sigma)) : \sigma\in\mathscr{E}_\mathscr{R}, \mathbf{x}\in\Omega\},$$

where

$$\mathscr{E}_\mathscr{R} = \bigcup \{\mathscr{K}_n(\varphi) : \varphi\in\rho_\mathbf{v}, \mathbf{v}\leqslant m\cdot\mathbf{1}\}.$$

Note that in general, $\mathscr{E}_\mathscr{R}$ need not be a triangulation in the above sense because adjacent simplices will no longer have to match up. Although therefore (3.7) cannot be applied directly it is an immediate consquence of (6.1) that still

$$\Pi_k(\Omega) \subset \mathscr{S}_k(H,\mathscr{R}) \tag{6.2}$$

holds. Indeed, (2.7), (6.1) and (5.9) affirm, that we have

$$\mathbf{x}^\alpha = \sum_{\pi,\nu} \xi^\alpha_{\pi,\nu} \left(\sum_{\sigma \subset H_\nu(\sigma_\pi)} \operatorname{vol}_n(\sigma)\, M(\mathbf{x} \mid P_\sigma) \right), \qquad \mathbf{x} \in \Omega. \tag{6.3}$$

Furthermore, note that the "finer" knot sets $P(\sigma)$ differ from the original ones in $\mathscr{P}_H$ only by dilation. Hence, (cf. the remarks in Section 2) the B-splines arising from finer knot sets have still the same smoothness as those generating the uniform space, i.e., in particular we may state for any refinement $\mathscr{R}$

$$\mathscr{S}_k(H, \mathscr{R}) \subset C^{k-1}(\Omega) \qquad \text{iff} \quad \mathscr{S}_k(H) \subset C^{k-1}(\Omega) \tag{6.4}$$

which in turn is equivalent to H being dispersive (cf. Theorem 4.1). Note also that for any refinement $\mathscr{R}$ the evaluation of a spline in $\mathscr{S}_k(H, \mathscr{R})$ essentially still involves (up to shifting and rescaling) only the evaluation of the B-splines induced by $\mathscr{K}_n(Q)$.

These local refinements will be used in a forthcoming paper to construct smooth adaptive approximation shemes for any spatial dimension and degree k which are still equivalent (as for the complexity of the involved spaces and approximation rates) to optimal adaptive piecewise polynomial approximation (cf. [6]).

Acknowledgment

The author would like to thank Dr. Charles A. Micchelli for many helpful discussions and suggestions while preparing this paper.

References

1. E. Allgower and K. Georg, Triangulations by reflections with applications to approximations, *in* "Proceedings, Conf. Oberwolfach, Numerische Methoden der Approximationstheorie Birkhäser, 1978."
2. I. Babuska, The self-adaptive approach in the finite element method, *in* "The Mathematics of the Finite Elements and Applications *II*: MAFELAP 1975" (J. R. Whiteman, Ed.), Academic Press, New York/London, 1976.
3. C. de Boor, On uniform approximation by splines, *J. Approx. Theory* **1** (1968), 219–235.
4. C. de Boor, Splines as linear combinations of B-splines, *in* "Approximation Theory II" (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), pp. 1–47, Academic Press, New York, 1976.
5. C. de Boor and G. Fix, "Spline approximation by quasi-interpolants, *J, Approx. Theory* **7** (1973), 19–45.
6. C. de Boor and J. R. Rice, An adaptive algorithm for multivariate approximation giving optimal convergence rates, *J. Approx. Theory* **25** (1979), 337–359.

7. J. H. BRAMBLE AND M. ZLAMAL, Triangular elements in the finite element methods, *Math. Comp* **24** (1970), 809–820.
8. M. G. COX, The numerical evaluation of B-splines, *J. Inst. Maths. Appl.* **10** (1972), 134–149.
9. W. DAHMEN, R. DEVORE, AND K. SCHERER, Multi-dimensional spline approximation, *SIAM J. Numer. Anal.* **17** (1980), 179–191.
10. W. DAHMEN, On multivariate B-splines, *SIAM J. Numer. Anal.* **17** (1980), 179–191.
11. W. DAHMEN, Polynomials as linear combinations of multivariate B-splines, *Math. Z.* **169** (1979), 93–98.
12. W. DAHMEN, Multivariate B-splines, recurrence relations and linear combinations of truncated powers, *in* "Proceedings, Conf. Oberwolfach, Multivariate Constructive Function Theory, Birkhäuser, 1979."
13. W. DAHMEN, Konstruktion Mehrdimensionaler B-splines and ihre Anwendungen auf Approximationsprobleme, *in* "Proceedings, Conf. Oberwolfach, Numerische Methoden der Approximationstheorie, Birkhäuser, 1979."
14. J. A. GREGORY AND J. R. WHITEMAN, "Local Mesh Refinement with Finite Elements for Elliptic Problems," Technical Report TR 124, Department of Mathematics, Brunel University, 1974.
15. H. JOHNEN AND K. SCHERER, On the equivalence of the K-functional and the moduli of continuity and some applications, *in* "Constructive Theory of Functions of Several Variables," pp. 119–140, Lecture Notes in Mathematics No. 571, Springer-Verlag, New York/Berlin, 1977.
16. S. KARLIN, "Total Positivity," Stanford Univ. Press, Stanford, Calif., 1968.
17. H. W. KUHN, Some combinatorial lemmas in topology, *IBM J. Res. Develop.* **45** (1960), 518–524.
18. T. LYCHE AND L. L. SCHUMAKER, Local spline approximation methods, *J. Approx. Theory* **15** (1975), 294–325.
19. C. A. MICHELLI, A constructive approach to Kergin interpolation in R^k: Multivariate B-splines and Lagrange interpolation," Mathematical Research Center, University of Madison, Wisconsin, Technical Summary Report 1978, *Rocky Mountain J. Math.* **10** (1980), 485–497.
20. J. MORGAN AND R. SCOTT, The dimension of the space of C^1 piecewise polynomials, unpublished.
21. C. B. MORREY, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin/Heidelberg/New York, 1966.
22. M. J. D. POWELL, Piecewise quadratic surface fitting for contour plotting, *in* "Proceedings, Conf. Software for Numerical Mathematics, University of Laughborough, 1973," pp. 253–271.